What math
knowledge does teaching require?

By Mark Hoover Thames and Deborah Loewenberg Ball

Effectively helping others learn is demanding work that necessitates sensibility as well as specialized knowledge and skill.

No one would argue with the claim that teaching mathematics requires mathematics knowledge. However, a clear description of such knowledge needed for teaching has been surprisingly elusive (Hill and Ball 2008). To differentiate teachers’ levels of mathematical knowledge, numerous studies have examined whether a teacher has a certification in math or a degree as well as the number of math courses taken. But analyses of the correlations between these indicators and students’ achievement gains reveal no advantage at the grades K–8 level and only slight advantage at the secondary level. These studies, carried out over the past forty years, do not contradict the assertion that mathematical knowledge matters for teaching math, but they do suggest that conventional content knowledge is insufficient for skillfully handling the mathematical tasks of teaching. Although it seems that majoring in math should provide an edge in teachers’ capacity, it simply does not at the grades K–8 level, and it is an uneven predictor at the high school level.

So what do teachers need? Intrigued by the problem of identifying the mathematical knowledge and skill that actually contribute to student learning, we and our colleagues at the University of Michigan directly studied the work of teaching to uncover the mathematical issues that arise in practice. Our conjecture was that by better understanding the mathematical questions and situations with which teachers must deal, we would gain a better understanding of the mathematics it takes to teach.

Over the course of several years, we observed and videotaped teaching in many different classrooms. We set out to identify common teaching tasks, and as we did, we began to see more clearly the mathematical demands of everyday teaching. We saw the math understanding involved in posing questions, interpreting
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Figure 1

Although the textbook did not suggest it, this teacher used a hundred chart to help her students learn near-ten strategies.

To highlight what our approach yielded, we turn to a lesson in a second-grade classroom where the teacher, Ms. Nash, is using the hundred chart (see fig. 1) to help her students learn near-ten strategies. This constructed vignette was inspired by a video (Rowland et al. 2009).

Nash repeats Rhonda’s answer, “So, to add nine to forty-six, you add ten and take away one. Can you show that on the hundred chart?” She offers Rhonda two red magnetic counters, having earlier modeled the process of placing both counters on the starting number and moving one counter to show the steps of the near-ten addition.

Nash asks Corey to summarize: “Do you think you can explain how we add nine and add eleven?” Corey comes to the front of the class, and explains, “You add ten, and then you take away or add one.”

Nash asks for clarification. “OK. What do you do if you add nine? Do you take away or add?” Corey responds correctly, “Take away.”

Mathematics teaching involves building students’ trust, managing behavior, and structuring time and space in ways that are conducive to learning. This requires both pedagogical know-how and interpersonal skills. However, as we see in the episode described above, teaching also makes mathematical demands of the teacher. Some of these demands are predictable, but others are less obvious. What is a near-ten strategy, and what is the mathematical point of teaching it? Is the purpose for students to master the students’ answers, providing explanations, and using representations. We heard it in teachers’ talk and in the language they taught their students to use. We realized that the capacity to see mathematical ideas from another’s perspective and to understand what another person is doing involves mathematical reasoning and skill not needed for research mathematics or for bench science.

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This second-grade class has just begun a lesson on adding near-ten numbers (e.g., 9, 11, 19, and 21) by adding a multiple of ten and then adjusting by one. The textbook had not suggested using a hundred chart, but Nash had been using one with her class recently and decided it would help them understand the near-ten strategy.

Having completed several examples, Nash asks Corey to summarize: “Do you think you can explain how we add nine and add eleven?” Corey comes to the front of the class, and explains, “You add ten, and then you take away or add one.”

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Mathematics teaching involves building students’ trust, managing behavior, and structuring time and space in ways that are conducive to learning. This requires both pedagogical know-how and interpersonal skills. However, as we see in the episode described above, teaching also makes mathematical demands of the teacher. Some of these demands are predictable, but others are less obvious. What is a near-ten strategy, and what is the mathematical point of teaching it? Is the purpose for students to master the
strategy or to increase students’ flexibility with computation? Another mathematical issue is to evaluate the appropriateness of the hundred chart. In what ways might this chart support or interfere with the main mathematical point? How might the “wrap-around” of the numbers at the end of each line be handled? Might the chart encourage “diagonal moves” that could be at odds with the place-value emphasis of a near-ten strategy? A third challenge is to determine what mathematics is at the heart of the lesson and to be able to see what aspects of this might be complex for learners. Here, being familiar with children is helpful, but mathematical insight matters, too. What would count as evidence that children understand this particular mathematics and will be successful in using it in the future?

Teachers face such “teaching problems” as these every day. Solving them demands mathematical understanding and flexibility. Beyond being well versed in the content of the curriculum, teachers need significant mathematical skill, perspective, and judgment. For instance, teaching requires being able to answer children’s “why” questions: Why do we find common denominators when adding fractions but not when multiplying them? Why do we count the places to the right of the decimal point and add them when we multiply decimals? Our studies of teaching have led us to identify specific tasks of teaching that require mathematical skill. The more we examine teaching, the more we find that teaching well requires an abundance of mathematical skill and of usable mathematical knowledge—mathematical knowledge in and for teaching. Consider, for instance, some of the most frequent tasks of teaching:

- **Posing** mathematical questions
- **Giving** and appraising explanations
- **Choosing** or designing tasks
- **Using** and choosing representations
- **Recording** mathematical work on the board
- **Selecting** and sequencing examples
- **Analyzing** students’ errors
- **Appraising** students’ unconventional ideas
- **Mediating** a discussion
- **Attending** to and using math language
- **Defining** terms mathematically and accessibly
- **Choosing** or using math notation

Based on our analysis of these tasks, we began to see that mathematical knowledge for teaching (MKT) consists of distinguishable domains, each defined in relation to the work of teaching. **Figure 2** represents all the mathematical knowledge important for teaching. One key point is that some of the mathematical resources that teaching requires are similar to the mathematical knowledge used in settings other than classrooms—being able to do particular calculations, knowing the definition of a concept, or making a simple representation, for example. We call this **common content knowledge** (see the top left of fig. 2); it is important for such teaching tasks as knowing whether a student’s answer is correct, providing students with the definition of a concept or an object, and demonstrating how to carry out a procedure.

Additionally, teachers also need a kind of knowledge that blends content knowledge with pedagogical knowledge, **pedagogical content knowledge**, or PCK (Shulman 1986, 1987; Wilson, Shulman, and Richert 1987). We have subdivided PCK into subdomains that combine knowledge of content with knowledge of students, teaching, and curriculum (see the right side of fig. 2).
However, some mathematical knowledge that teaching entails is specialized and not used by others—for example, being able to use the hundred chart to model specific aspects of place value, defining terms in mathematically correct but accessible ways, or being able to make sense of solutions other than one’s own. Each of these represents a mathematical task that can be thought about independent of specific students, teaching, or curriculum. We have also become interested in what we call horizon knowledge, a domain of MKT that affords a kind of mathematical “peripheral vision” needed in teaching (Ball and Bass 2009). For instance, in the previous episode, Nash is not teaching the standard addition and subtraction algorithms, but she should probably have them in mind and see potential connections. Perhaps, also, knowing something about different base number systems sheds a certain light on the mathematical work being done, but as an added perspective, without its becoming something to teach.

To extend this picture of the mathematical demands of teaching and to see how teacher content knowledge can support high-quality instruction, we return to our second-grade lesson, in which Nash, confident of students’ understanding of near-ten addition, introduces subtraction. We sketch two scenarios of how the lesson might play out: one in which teaching and learning go awry and a second in which usable teacher knowledge supports success.

Finding mathematical pitfalls of mathematics teaching

Nash puts two red markers on the square with the 70 and calls on Chad: “How could we find the answer to seventy minus nineteen?”

He answers, “You want to take away twenty and then take away one.”

Nash repeats his statement but then corrects it. “Not for nineteen. We take away twenty [she moves the counter from the 70 to the 50], which is fifty, but then we add one [she moves the counter to the 51], because we only wanted to take away nineteen, not twenty. So the answer is fifty-one.”

Nash has students count back to verify the answer, anticipating that this will lead them to recognize that subtracting nineteen is accomplished by subtracting twenty and then adding one. She reinforces the idea with a second example, seventy minus nine. However, students fumble, guessing at whether to add one or subtract one. When she asks, “How do I take away nine? What do you do first?” Rebecca replies, “Go up one—take away one.”

Nash explains again: “Take away one to take away nine? No. Remember when we added nine, we added ten first of all, so what do you think we might take away here? Simon?”

Simon responds, “Ten.”

“Take away ten; take away ten gives us sixty [she moves the counter from the 70 to the 60], and then what must we do when we’re taking away? We’re taking away nine here, so we’re taking away ten and…?”

[Simon answers, “Add one,” but he says it hesitantly.]

Nash continues: “Add one [moving the counter from the 60 to the 61]. Which brings us to sixty-one. Right?”

Sensing the students’ need for more explicit support, Nash makes a table on the white-
board (see fig. 3), organizing the four cases and giving examples for students to use as they complete the activity sheet for the lesson. Students seem unsure, but they have done several examples together and have the table to guide their work. Nash distributes the activity sheet and sets the students to work, but she seems frustrated and concerned that the children will have difficulty. She seems to have a sense that something is amiss but is unsure what it is or how to recover.

Using mathematical knowledge in and for teaching

Now imagine instead that Nash considers the mathematical conventions and the everyday conventions that associate “up” and “to the right” with increase. Imagine that she decides to replace the original hundred chart with one that is oriented “upside down” (see fig. 4) to align better with the language of addition (“more”) and subtraction (“less”) with “up,” “down,” “right,” and “left.” (This idea was suggested by Hyman Bass and developed with Meghan Shaughnessy and Nicole Louie in the Elementary Mathematics Laboratory at the University of Michigan.)

Nash continues with subtraction, posing forty-five minus nineteen. She puts two red markers on the 45 on the hundred chart and calls on Chad. He says, “You want to take away twenty and take away one.”

Nash asks Chad to come up and explain his thinking by using the hundred chart. He moves one of the markers down two rows and then moves it to the left, repeating his statement that you first take away twenty and then you take away one, but Nash interjects, “Why do you subtract twenty?”

Chad pauses, then says, “Because twenty is close to nineteen. It’s easy to subtract twenty.”

Nash continues, “And how do you decide whether to add or to subtract one?” Chad replies that you subtract one because it is a subtraction problem.

Nash recognizes that—of all the different problems—this is the most difficult one to reason about: The process of first subtracting twenty and then adding one as a way to “undo” having already subtracted too many is not easy for children to understand. It is mathematically complex. Arithmetically, it requires “distributing” subtraction across the parenthetical expression:

\[ 45 - 19 = 45 - (20 - 1) = 45 - 20 + 1 \]

Nash pauses before addressing the class:

I want everyone thinking hard about this. [She writes \( 45 - 19 = \) on the board.] In order to subtract nineteen, Chad says you first subtract twenty because it is close to nineteen. Do you agree? [Several students respond affirmatively.] What do we do next? Do we subtract one or add one?

Nash again pauses briefly before she continues. “Remember when we were adding nine,
Rhonda explained that you add ten and then subtract one? Who remembers her explanation? 
Nate recalls that Rhonda explained that you subtract one because “you don’t want to add so many as ten, so you have to take one away.”
Nash repeats Nate’s explanation:

Yes, Rhonda said that you need to subtract one—because you added ten but you only want to add nine. You don’t want to add so many. Remember how she showed us on the hundred chart how you want to move to the left because when you move up you added ten, which means you went too far? I want everyone to think about how to subtract nineteen from forty-five on the hundred chart. First you subtract twenty. Then what do you do? Talk with your partner.

While the children talk, Nash listens to their conversations. After a few minutes, she calls the class back together to discuss the ideas. Salvador goes to the hundred chart to explain his thinking. “I think you go to the right because when you go down two [pointing at the 25], that’s too many; that’s twenty. When you go down two, you take away all of these [pointing to the numbers from 45 back to 25]. That’s twenty, so that’s too many.

Nash asks if someone can explain Salvador’s thinking. Carl explains that Salvador said you have to go to the right “because twenty-five is twenty less than forty-five, so you want to add one back in.”

Jamie agrees, “I think you add one because if you subtract one then you’d be subtracting twenty-one.”

Nash writes the subtraction problem on the board and asks students how she could record the steps Salvador used. The children suggest using arrows, and she records this as 45, two down arrows, and one arrow to the right (see fig. 5). She then asks for another way to write the two down arrows “using numbers and addition and subtraction.” Students decide that the two down arrows would mean “subtract twenty” and that the right arrow would mean “add one.” The class agrees that you end at the 26.

In this second scenario, Nash draws on significant mathematical knowledge and insight to help her manage instruction and direct children’s learning. She recognizes the mathematical challenge in compensating for having subtracted too many. She also recognizes the mathematical problems with the representation of “up” for subtraction in the original hundred chart. She attends to the inadequacy of Chad’s initial explanation and focuses children’s attention on this key mathematical issue for the lesson. And she recognizes the complexity involved in reasoning that the extra “one” must be added, not subtracted, and the need to return students to Rhonda’s analogous reasoning about the need to subtract one when adding nine. In this second scenario, Nash exhibits mathematical knowledge and skill in sizing up the mathematical issues in math problems, in managing mathematical talk in the classroom, and in attending to the mathematical basis for explanations that support understanding. Far from being straightforward, effective teaching involves significant, specialized mathematical knowledge and skill.

**Strengthening mathematical capacity in practice**

Consider the following real-world teaching problem and some possible solutions (expressed as a multiple-choice assessment problem). Nash is teaching a lesson on a near-ten subtraction strategy in which, to subtract numbers close to a multiple of ten, you subtract the multiple of ten, then compensate with addition or subtraction. For example, to subtract nineteen or twenty-one, subtract twenty, then add or subtract one. She asks her class to use the “upside-down” hundred chart (see fig. 4) to solve the problem forty-five minus nineteen. Students agree the answer is twenty-six but offer different reasoning. Of the following student explanations, which uses the near-ten
strategy and gives the best mathematical basis for why it works?

1. It's twenty-six because you start at 45 and go down two rows because each row is ten. So two rows is twenty, and then you go to the right one step because each time you move to the right, it's adding one.

2. It's twenty-six because 26 is down two and one over, and if you count all the numbers on the chart from 26 up to 45, you get nineteen.

3. It's twenty-six because you go down two rows to subtract twenty, but you've subtracted too many, so you have to add one back in.

4. All these explanations offer an equally good mathematical basis for why the near-ten strategy works.

All these student explanations make sense, but a teacher must recognize when key mathematical issues are being addressed and when they are being missed. In this case, why is subtracting twenty, then adding one equivalent to subtracting nineteen, and how do you coordinate the hundred chart to explain this? The first explanation identifies the rule of subtracting二十 and adding one and coordinates the hundred chart with the rule, but it fails to explain why one should be added rather than subtracted. The second explanation points out that the number 26 is down two squares and over one square to the right, but it justifies the answer of twenty-six independent of the near-ten strategy. The last explanation might be elaborated on, but it explains the key step of why you add one in the context of this subtraction problem.

Knowing how to respond to this problem requires mathematical knowledge not typically taught in mathematics courses or in workshops for teachers and not typically assessed when measuring teacher knowledge.
It is a type of mathematics that arises in and for teaching. It is this sort of math knowledge that we argue needs to be better understood.

Teaching is not merely about doing math oneself, but about helping students learn to do it. This is challenging and requires specialized, skilled ways of knowing the domain. Examining practice itself—from planning lessons to using textbooks, leading a discussion, using the board carefully, and choosing examples—reveals the mathematical demands of the work, which are often overlooked. Identifying these demands allows us to identify the mathematical knowledge needed for teaching.

Appreciating the mathematical work of teaching is crucial to being able to support teachers in that work. For example, teachers’ opportunities to learn mathematics could be better tied to actual situations that come up in teaching. Teachers could practice dealing with the mathematical problems that arise every day and could develop the mathematical skill and fluency needed in practice. Teacher education and professional development could center more directly on the mathematical knowledge on which effective teaching draws.

There are signs of promise: In recent studies of professional development, programs that connect the mathematical content to teaching practice produced greater gains in teachers’ mathematical knowledge for teaching and in their students’ achievement than programs that focused merely on content knowledge. (See, for example Cohen and Hill 2001; Hill and Ball 2004; and Gross, Harris, and Meyers forthcoming.)

Mathematical knowledge does matter for teaching. But it is not a mathematical expertise like that required for research in mathematics or for other kinds of quantitative work. Instead, mathematical knowledge for teaching is a kind of complex mathematical understanding, skill, and fluency used in the work of helping others learn mathematics.

BIBLIOGRAPHY


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